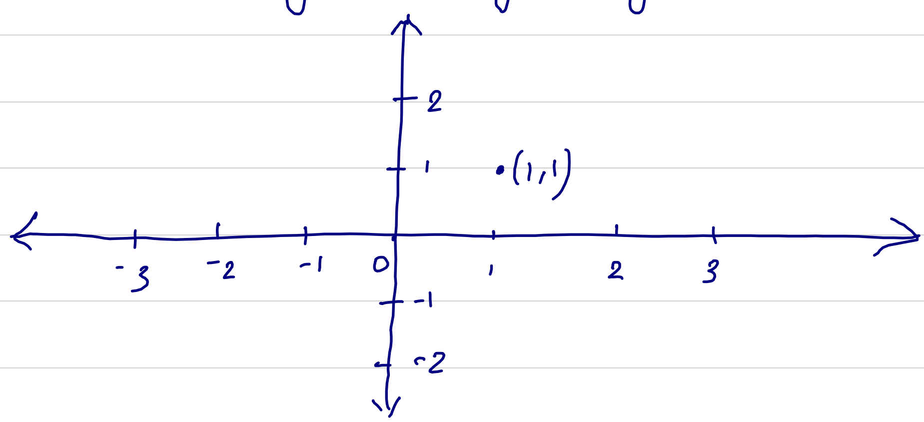
Section 1.2

Cartesian plane

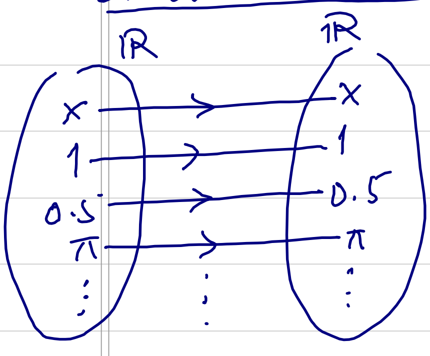
Named after Rene Descartes. We have two real lines intersecting at a right angle.



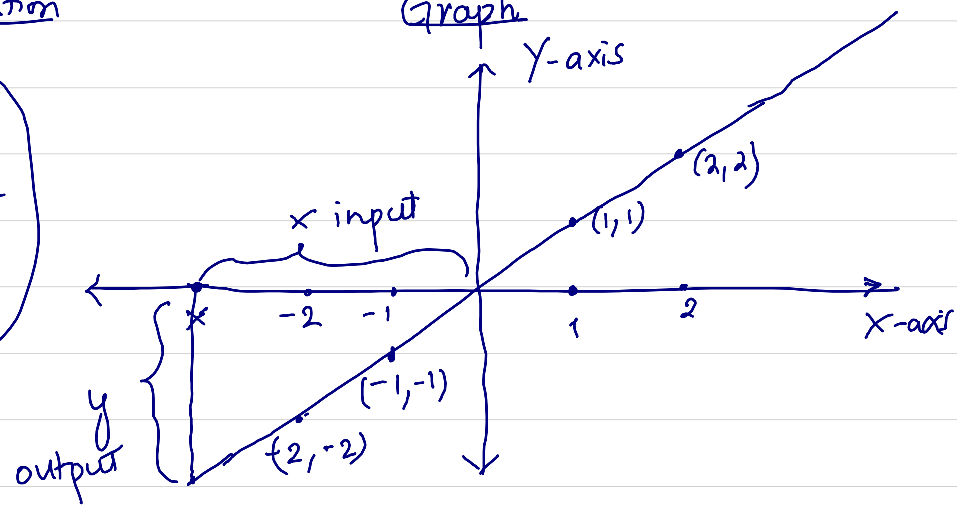
Using coordinates, i.e. ordered pair of numbers like (1,1), we can locate any point in the plane. We use the Cartesian plane to graph functions. The inputs will go in the X-axis or the horizontal line. The respective outputs go directly above the inputs.

Ex: The function $f: \mathbb{R} \to \mathbb{R}$ defined as $y = x$ or $f(x) = x$

In oval set notation



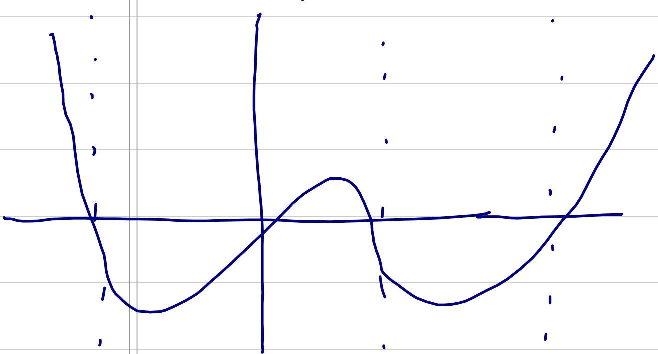
Graph



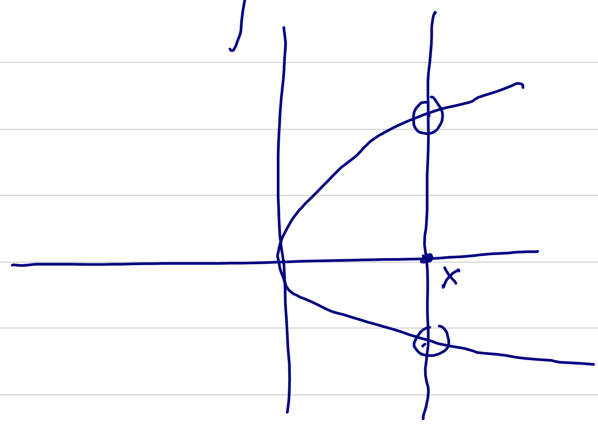
How do you test whether a certain graph is a graph of a function?

Vertical line Test

"Any vertical line in the plane intersects the graph of a function at at most one point.



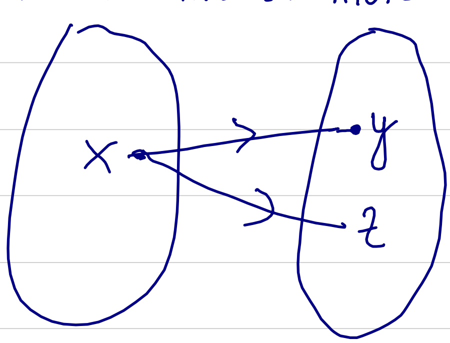
Graph of a functn



Not a function. x has two different outputs.

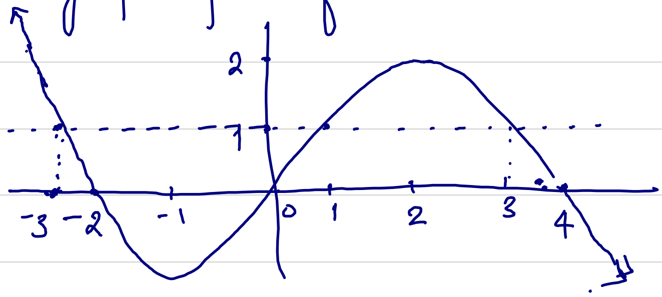
Why does this work?

All this test is saying is that each input must have only one output. If a line intersects the graph at two or more points, it implies that one input has two or more outputs. Pictorially



This cannot happen in a function!

Given the graph of the function



Find: (a) $f(0) = 0$

(b) $f(2) = 2$

(c) Solve $f(x) = 0$, i.e. find x such that $f(x) = 0$.

Solution. $f(-2) = 0$

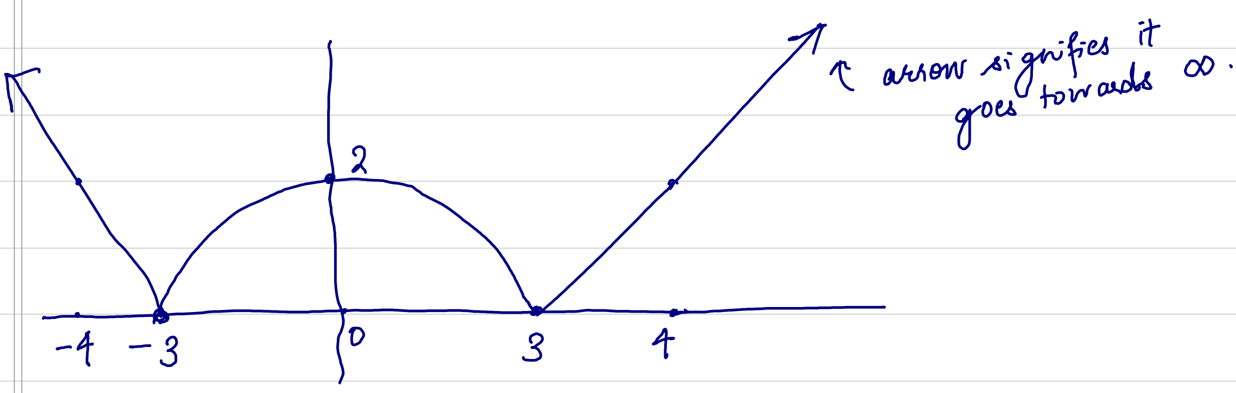
$f(0) = 0$ Thus $x \in \{-2, 0, 4\}$

$f(4) = 0$

(d) Solve $f(x) = 1$

$x = -3, 1$ or 3.

EXERCISE 1



(a) Is the graph a function?

(b) Find the domain of the function.

(c) Find the range of the function.

(d) Find all x such that $f(x) = 0$.

(e) Solve $f(x) = 2$.

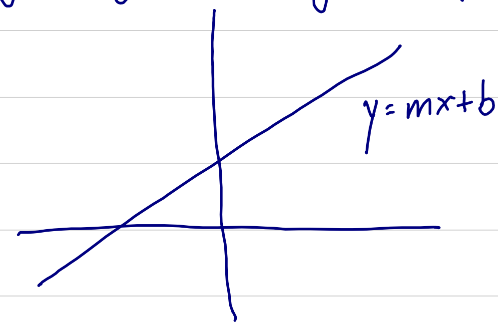
Common Functions

Linear Function

$$y = mx + b$$

m gives you the slope

b gives you the y-intercept / vertical intercept



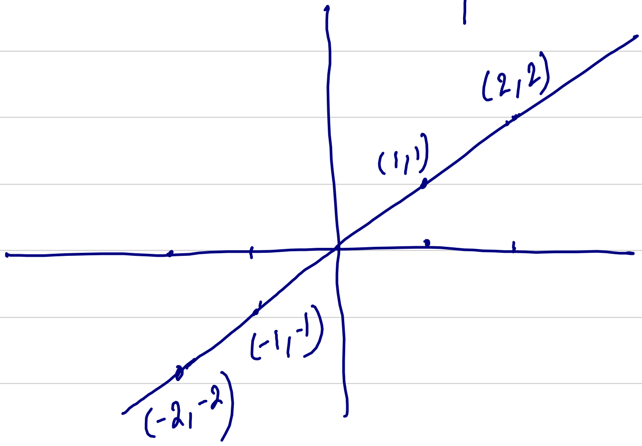
Graph is always a line

(i) Special case: $m = 0$, then

$$y = b \quad \text{(constant function)}$$

(ii) Special case: $m=1$, $b=0$, then

$$y = x$$



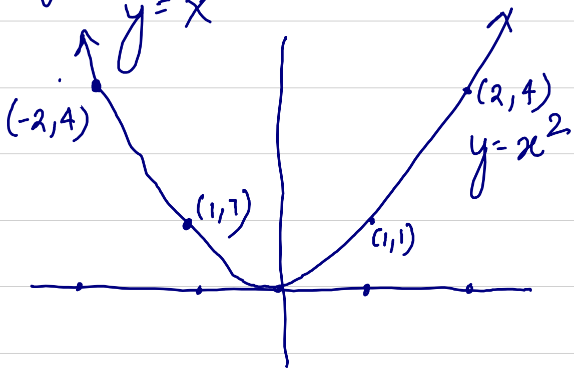
x	y
1	1
2	2
-1	-1
-2	-2

Domain = $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Square function

$y = x^2$



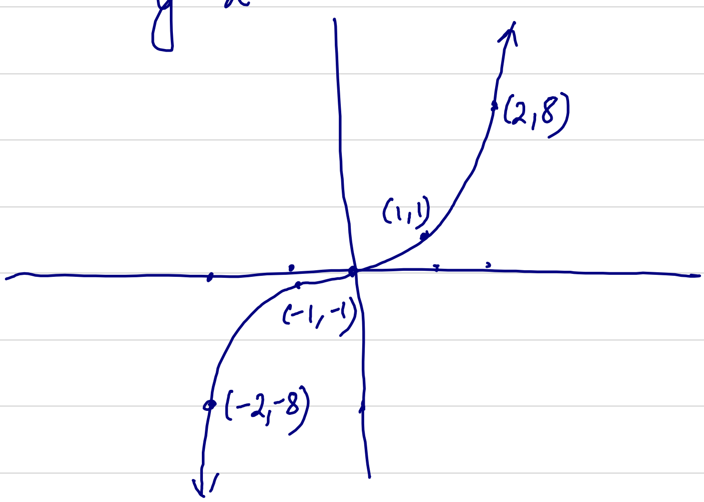
x	y
1	1
2	4
-1	1
-2	4

Domain = $(-\infty, \infty)$

Range = $[0, \infty)$

Cube function

$y = x^3$



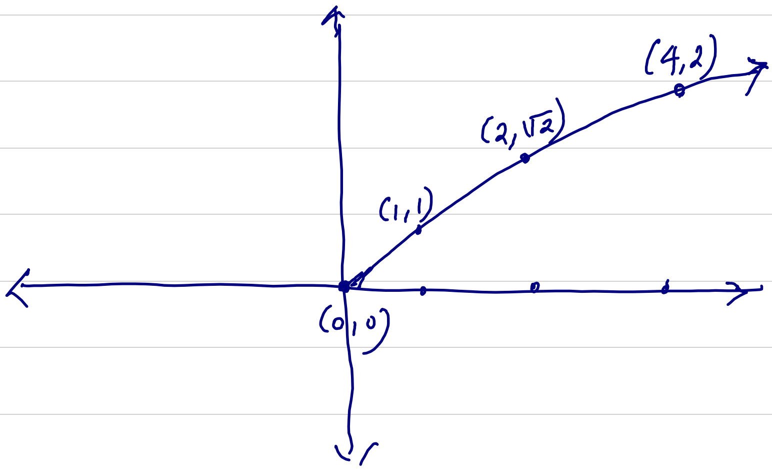
x	y
1	1
2	8
-1	-1
-2	-8

Domain = $(-\infty, \infty)$

Range = $(-\infty, \infty)$

Square root function

$f(x) = \sqrt{x}$ or $x^{\frac{1}{2}}$



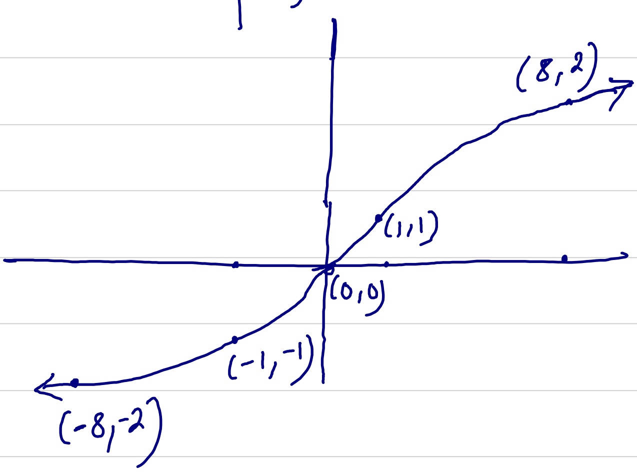
x	y
1	1
2	$\sqrt{2}$
4	2
9	3
0	0

Domain = $[0, \infty)$

Range = $[0, \infty)$

Cube root function

$f(x) = \sqrt[3]{x}$ or $x^{\frac{1}{3}}$



x	y
1	1
8	2
-1	-1
-8	-2
0	0

Domain = $(-\infty, \infty)$

Range = $(-\infty, \infty)$

Absolute Value Function

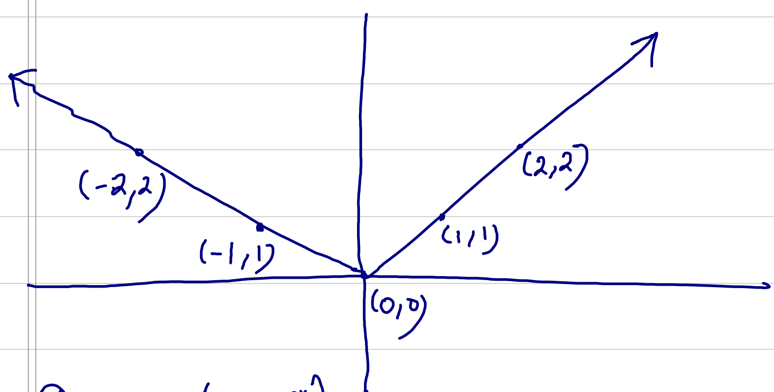
· $f(x) = |x|$

Def.: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

X	Y
1	1
0	0
2	2
-1	1
-2	2

Ex: $|4| = 4$, $|0| = 0$, $|-2| = 2$.

It is the distance of the number from 0 which is always positive except at 0.



Domain = $(-\infty, \infty)$

Range = $[0, \infty)$.

Reciprocal Function

$$f(x) = \frac{1}{x}$$

We know that Domain = $(-\infty, 0) \cup (0, \infty)$

Ques. What happens if I plug in a really large number for x?

Ans. $f(\text{very large}) = \dfrac{1}{\text{very large}} = \text{very small}$

For instance, $f(100000) = \dfrac{1}{10000}$

very large

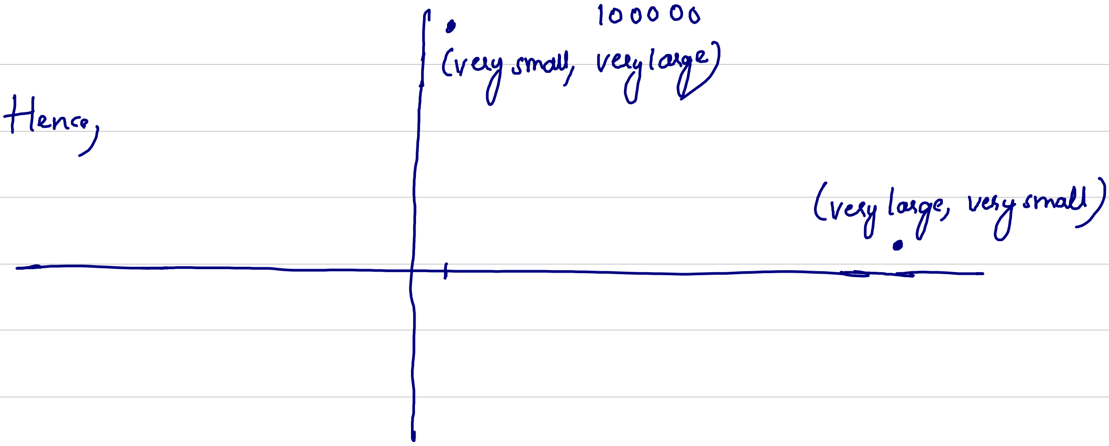
very small

<u>Ques</u> What happens if I plug a small number for x?

<u>Ans.</u> $f(\text{very small}) = \dfrac{1}{\text{very small}} = \text{very large}$

For instance, $f\left(\dfrac{1}{100000}\right) = \dfrac{1}{\frac{1}{100000}} = 100000$

Hence,



Similarly, for negative numbers,

$$f(-\text{very small}) = \frac{1}{-\text{very small}} = -\frac{1}{\text{very small}}$$
$$= -\text{very large}$$

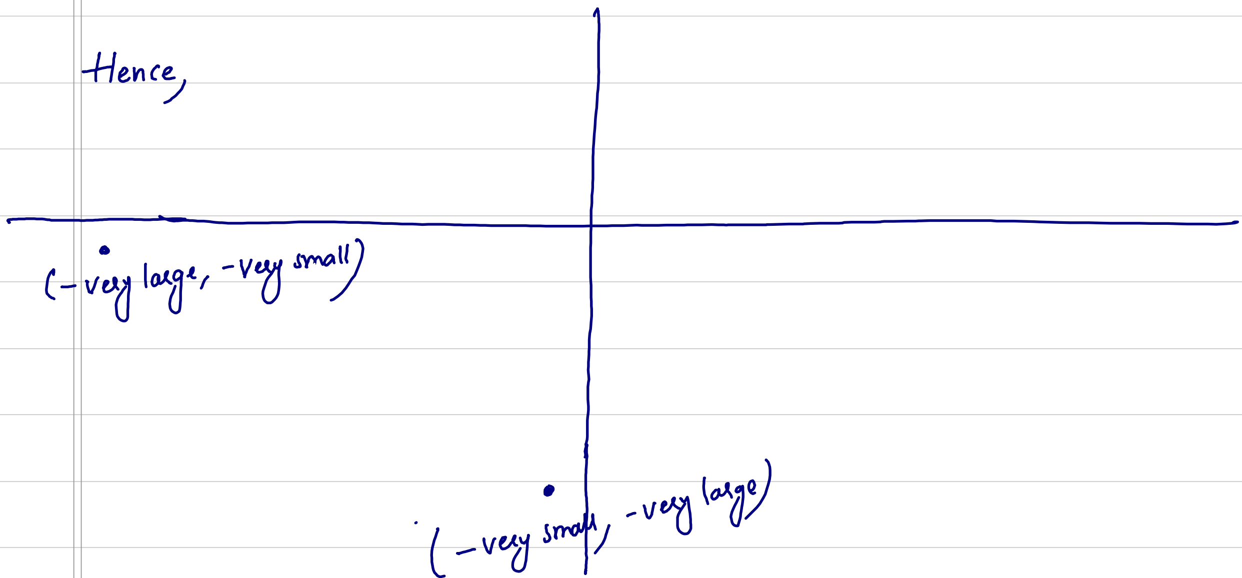
For instance, $f\left(-\dfrac{1}{100000}\right) = -\dfrac{1}{\frac{1}{100000}} = -100000$

And,

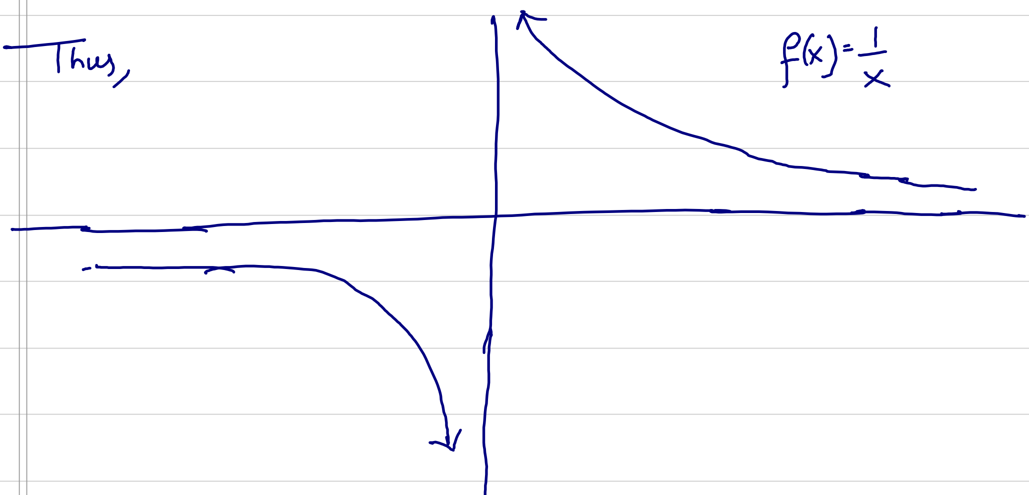
$$f(-\text{very large}) = \frac{1}{-\text{very large}} = -\frac{1}{\text{very large}}$$
$$= -\text{very small}$$

For instance, $f(-1000000) = \dfrac{1}{-1000000} = -\dfrac{1}{1000000}$

Hence,



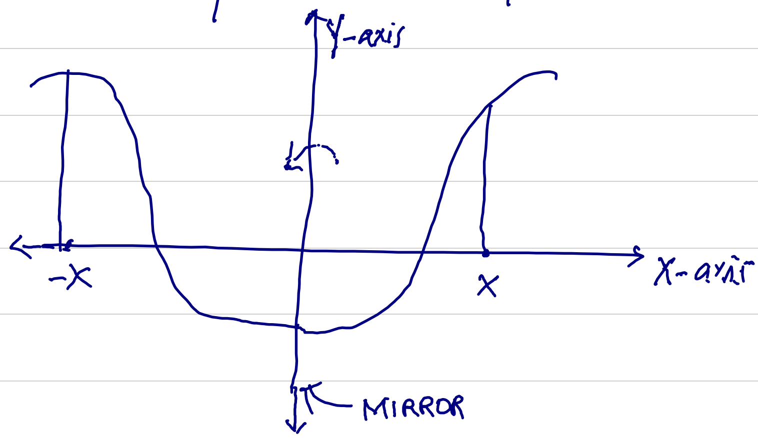
Thus,



<u>EXERCISE 2</u> Find the range of $f(x) = \frac{1}{x}$.

Even and Odd Functions

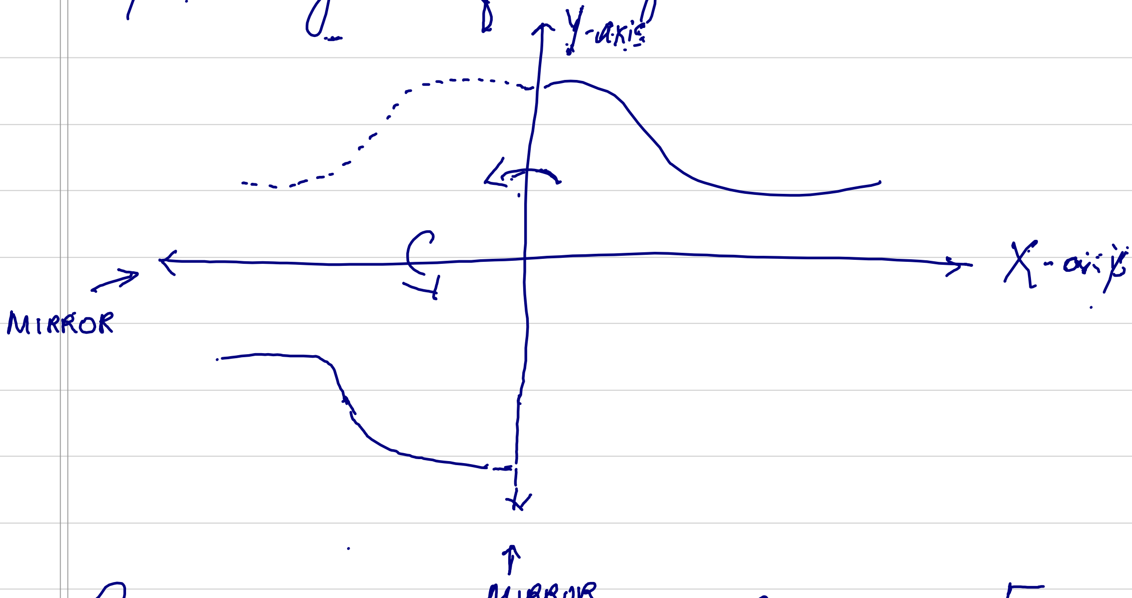
A function is <u>even</u> if $f(-x) = f(x)$, i.e., it is symmetric with respect to the Y-axis.



<u>Examples</u>: $f(x) = x^2$, $f(x) = x^4$, $f(x) = |x|$

A function is <u>odd</u> if $f(-x) = -f(x)$, i.e.

1) First you reflect along Y-axis.
2) Then you reflect along X-axis



<u>Examples</u>: $f(x) = x$, $f(x) = x^3$, $f(x) = \sqrt[3]{x}$

Determine whether the functions are even, odd, or neither

(a) $f(x) = x^2 - 3$

<u>Solution</u>.

$$f(-x) = (-x)^2 - 3$$
$$= x^2 - 3$$
$$= f(x)$$

Since $f(-x) = f(x)$, f is $\boxed{\text{even.}}$

(b) $g(x) = x^5 + x$

<u>Solution</u>

$$g(-x) = (-x)^5 + (-x)$$
$$= -x^5 - x$$
$$= -(x^5 + x)$$
$$= -g(x)$$

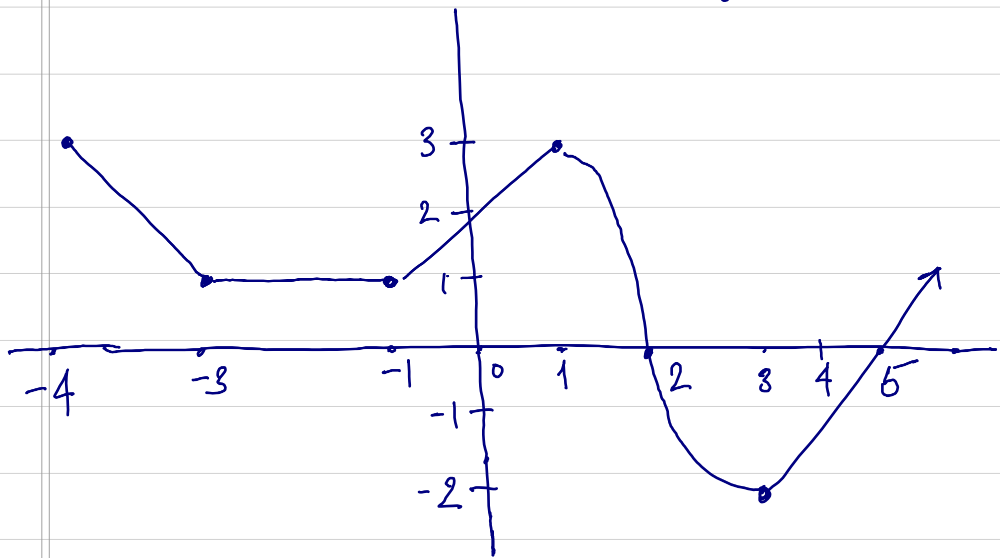
Since $g(-x) = -g(x)$, g is $\boxed{\text{odd}}$.

<u>EXERCISE 2</u> Classify the following functions as even, odd or neither

(a) $f(x) = |x| + 4$

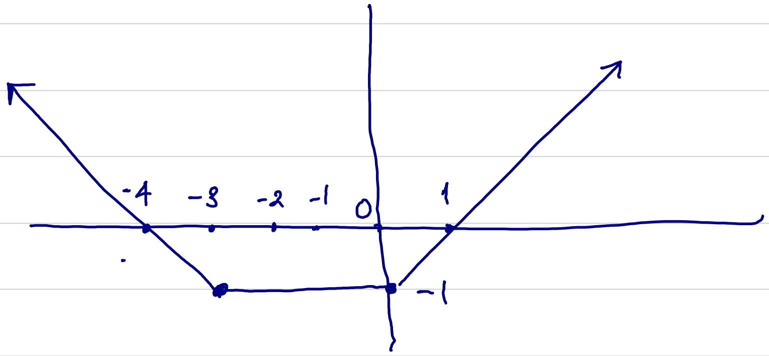
(b) $f(x) = x^3 - 1$.

Increasing, Decreasing, and Constant functions



Domain: $[-4, \infty)$
Range: $[-2, \infty)$
Increasing: $(-1, 1) \cup (3, \infty)$
Decreasing: $(-4, -3) \cup (1, 3)$
Constant: $(-3, -1)$

EXERCISE 3



Find (a) domain
(b) Range
(c) Intervals where it is (i) increasing
(ii) decreasing
(iii) constant

Average rate of change

We will need to learn Calculus to find the instantaneous rate of change at any value of a function. But we can find the average rate of change of a function using elementary methods.

Ques. Suppose we drove from LFT to IAH on a car. The whole trip is 220 miles. Say that it took us 4 hours to complete the trip. What was our average velocity?

Ans. $$\text{Average velocity} = \frac{\text{Total distance}}{\text{Total time taken}}$$

$$= \frac{220 \text{ miles}}{4 \text{ hours}}$$

$$= 55 \text{ miles/hr}$$

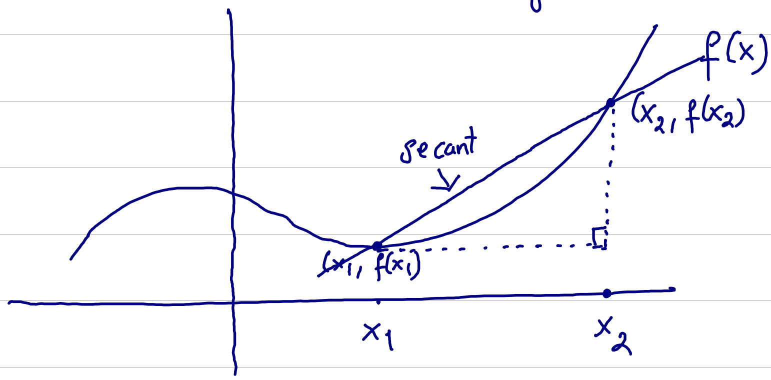
The above example calculated the average rate of change for the velocity function. But we can do the same for any function.

$$\boxed{\text{Average rate of change from } x_1 \text{ to } x_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

(where $x_1 < x_2$)

Note: x_1 and x_2 stand for numbers in the domain

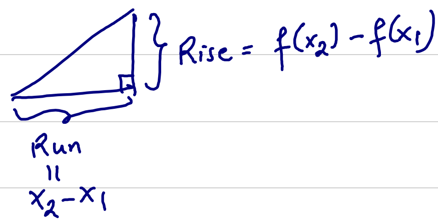
Geometric interpretation of the above formula:



Average rate of change

$= $ Slope of secant

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



<u>Example</u> Find average rate of change of $f(x) = x^2 - 1$ from $x_1 = -2$ to $x_2 = 3$

<u>Soln.</u> We know that

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{x_2^2 - 1 - (x_1^2 - 1)}{x_2 - x_1}$$

$$= \frac{3^2 - 1 - [(-2)^2 - 1]}{3 - (-2)}$$

$$= \frac{9 - 1 - [4 - 1]}{3 + 2}$$

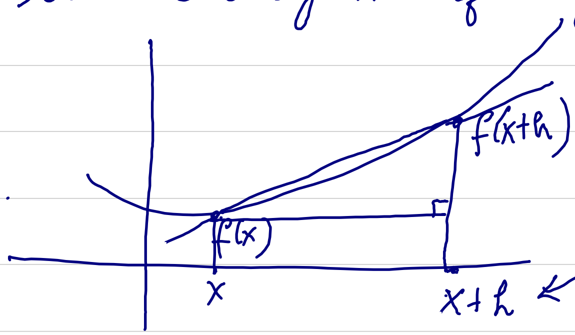
$$= \frac{8 - 3}{5}$$

$$= \frac{5}{5} = \boxed{1}$$

EXERCISE 4 Find the average rate of change of $f(x) = x^2$ from
(a) $x = -2$ to $x = 0$.
(b) $x = 0$ to $x = 2$.

<u>Difference Quotient</u>

Same as average rate of change. We are just changing the variables.



(We are writing x_2 as $x+h$ where h is some positive number)

$$\text{Difference quotient} = \frac{f(x+h) - f(x)}{(x+h) - x}$$
$$= \frac{f(x+h) - f(x)}{h}$$

This is how the <u>derivative</u> is defined in Calculus. You take the limit of this quantity as h goes to 0, and then you get the slope of the tangent.

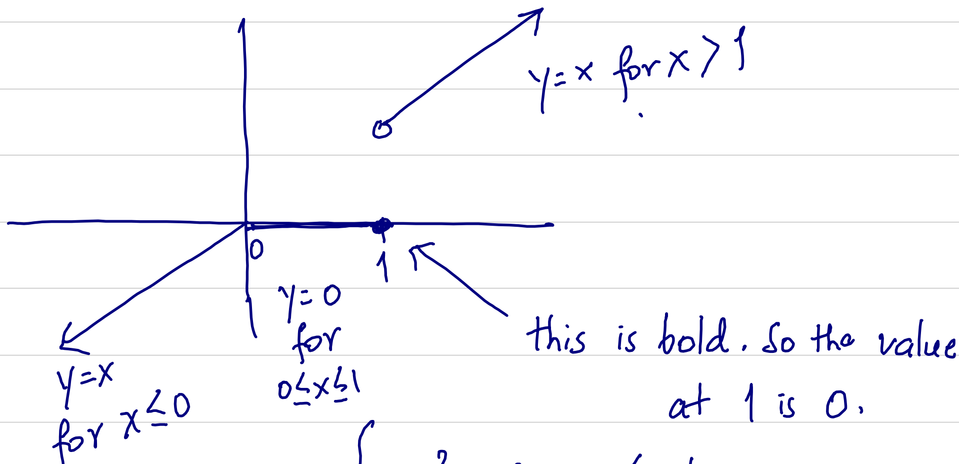
EXERCISE 5 Find the difference quotient for $f(x) = 2x^2 + 1$

EXERCISE 6 Find the difference quotient for $f(x) = -x^2 + 2$.

Piecewise defined functions

Sometimes you won't be able to define functions by a single formula like $f(x) = x^2$ or $f(x) = x^3$. You might have to define them piecewise.

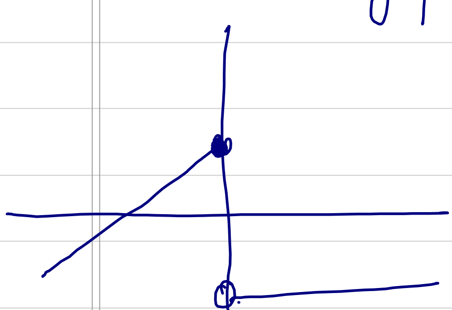
Ex.



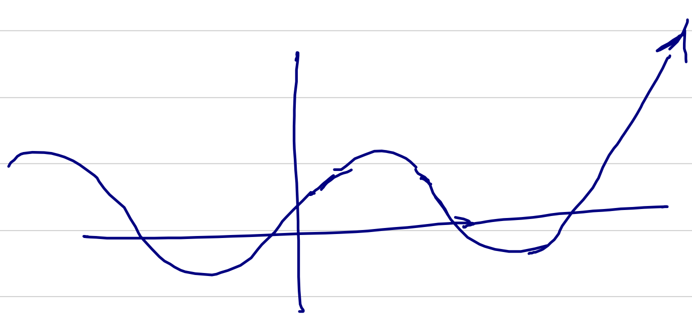
Exercise 7 Graph $f(x) \begin{cases} -x^2 & \text{for } x < -1 \\ x^3 & \text{for } -1 \le x < 0 \\ 1 & \text{for } x \ge 0 \end{cases}$

Continuity of functions

Took a while for mathematicians to define continuity. You will learn this is Calculus. But for now just understand that continuous means no gaps.



Discontinuous at 0



Continuous

EXERCISE 8

Graph the following function

$$f(x) = \begin{cases} x & x \le -1 \\ x^3 & -1 < x < 1 \\ x^2 & x > 1 \end{cases}$$

Then find the domain and range in interval notation. Also determine where the function is increasing, decreasing, or constant.